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A closed form for the Stirling polynomials in terms of the Stirling numbers

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Abstract

In the paper, by virtue of the Faá di Bruno formula and two identities for the Bell polynomial of the second kind, the authors find a closed form for the Stirling polynomials in terms of the Stirling numbers of the first and second kinds.

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1 Notation and main result

It is common knowledge [1, p. 48] that the Bernoulli numbers B_j are defined by

$$\frac{z}{e^z-1}=\sum_{j=0}^{\infty}B_j\frac{z^j}{j!}=1-\frac{z}{2}+\sum_{j=1}^{\infty}B_{2j}\frac{z^{2j}}{(2j)!},\quad |z|<2\pi.$$

For some new developments in recent years about this topic, please refer to [2, 3, 5, 9, 12] and the closely related references therein.

The Stirling numbers of the first and second kinds $s(n,k)$ and $S(n,k)$ are important in combinatorial analysis, theory of special functions, and number theory. They can be generated by the rising factorial

$$(x)_n=\prod_{k=0}^{n-1}(x+k)=\sum_{k=0}^{n}s(n,k)x^k \tag{1}$$

and the exponential function

$$\frac{(e^x-1)^k}{k!}=\sum_{n=k}^{\infty}S(n,k)\frac{x^n}{n!},$$

see [1, p. 213, Theorem A] and [1, p. 206, Theorem A], and can be computed by explicit formulas

$$s(n,k)=(-1)^{n+k}(n-1)!\sum_{\ell_1=1}^{n-1}\frac{1}{\ell_1}\sum_{\ell_2=1}^{\ell_1-1}\frac{1}{\ell_2}\cdots\sum_{\ell_{k-2}=1}^{\ell_{k-3}-1}\frac{1}{\ell_{k-2}}\sum_{\ell_{k-1}=1}^{\ell_{k-2}-1}\frac{1}{\ell_{k-1}}$$

for $1\le k\le n$ and

$$S(n,k)=\frac{1}{k!}\sum_{\ell=1}^{k}(-1)^{k-\ell}\binom{k}{\ell}\ell^n,$$

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see [11, Corollary 2.3] and [1, p. 206]. For recent investigations on the Stirling numbers of the first and second kinds $s(n,k)$ and $S(n,k)$, please refer to [5, 9, 10, 11] and plenty of references cited therein.

The Stirling polynomials $S_k(x)$ are a family of polynomials that generalize the Bernoulli numbers B_k and the Stirling numbers of the second kind $S(n,k)$. The Stirling polynomials $S_k(x)$ for nonnegative integers k are defined by the generating function

$$\left(\frac{t}{1-e^{-t}}\right)^{x+1}=\sum_{k=0}^{\infty}S_k(x)\frac{t^k}{k!}.$$

The first six Stirling polynomials $S_k(x)$ for $0\le k\le 5$ are

$$1,\quad \frac{x+1}{2},\quad \frac{3x^2+5x+2}{12},\quad \frac{x^3+2x^2+x}{8},$$
$$\frac{15x^4+30x^3+5x^2-18x-8}{240},\quad \frac{3x^5+5x^4-5x^3-13x^2-6x}{96}.$$

The Stirling polynomials $S_k(x)$ for $k\ge 0$ are special cases of the Nölund polynomials $B_k^{(x)}(z)$ defined by

$$\left(\frac{t}{e^t-1}\right)^{x}e^{zt}=\sum_{k=0}^{\infty}B_k^{(x)}(z)\frac{t^k}{k!},$$

namely, $S_k(x)=B_k^{(x+1)}(x+1)$. See [8, Chapter 6].

By the way, the polynomials $\psi_k(x)$ defined by

$$\left(\frac{t}{1-e^{-t}}\right)^{x+1}=1+(x+1)\sum_{k=0}^{\infty}\psi_k(x)t^{k+1}$$

are also called the Stirling polynomials in [6] and [7, p. 71].

We can easily check

$$S_k(0)=(-1)^kB_k \quad \text{and} \quad S_k(-m)=\frac{(-1)^k}{\binom{k+m-1}{k}}S(k+m-1,m-1)$$

for $m\ge 1$. Moreover, the explicit formulas

$$\begin{aligned}S_k(x)&=(-1)^k\sum_{j=0}^{k}(-1)^jS(k+j,j)\frac{\binom{x+j}{j}\binom{x+k+1}{k-j}}{\binom{k+j}{j}}\\&=\sum_{j=0}^{k}(-1)^js(k+j+1,j+1)\frac{\binom{x-k}{j}\binom{x-k-j-1}{k-j}}{\binom{k+j}{k}},\end{aligned}$$

which come from Lagrange's interpolation formula, are known. For more information on $S_k(x)$, see the papers [15, 16] and the closely related references therein.

A closed form is a mathematical expression that can be evaluated in a finite number of operations. It may contain constants, variables, four arithmetic operations, and elementary functions, but usually no limit.

In this paper, we find a closed form for the Stirling polynomials $S_k(x)$ in terms of the Stirling numbers of the first and second kinds $s(n,k)$ and $S(n,k)$.

Our main result can be stated as the following theorem.

Theorem 1. *For $k \ge 0$, the Stirling polynomials $S_k(x)$ can be computed by the closed form*

$$S_k(x) = (-1)^k k! \sum_{m=0}^{k} \left[\sum_{\ell=m}^{k} \frac{s(\ell+1, m+1)}{(k+\ell)!} \sum_{i=0}^{\ell} (-1)^i \binom{k+\ell}{\ell-i} S(k+i,i) \right] x^m. \tag{2}$$

2 Proof of Theorem 1

The Bell polynomials of the second kind, or say, partial Bell polynomials, denoted by $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ for $n \ge k \ge 0$, are defined by

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{\ell_1,\dots,\ell_n \in \{0\}\cup\mathbb{N} \\ \sum_{i=1}^{n} i\ell_i = n \\ \sum_{i=1}^{n} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}.$$

See [1, p. 134, Theorem A]. They satisfy

$$\mathrm{B}_{n,k}\big(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\big) = a^k b^n \mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \tag{3}$$

for complex numbers a and b, see [1, p. 135], and

$$\mathrm{B}_{n,k}\left(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-k+2}\right) = \frac{n!}{(n+k)!} \sum_{i=0}^{k} (-1)^{k-i} \binom{n+k}{k-i} S(n+i,i), \tag{4}$$

see [3, Theorem 1 and Remark 1], [4, p. 30], [10, p. 315], [12, Lemma 2.3], [14, Remark 2.1], and [17, Example 4.2].

The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}\, t^n} f \circ h(t) = \sum_{k=0}^{n} f^{(k)}(h(t)) \mathrm{B}_{n,k}\big(h'(t), h''(t), \dots, h^{(n-k+1)}(t)\big). \tag{5}$$

See [1, p. 139, Theorem C].

Taking $f(u) = u^{-(x+1)}$ and $u = h(t) = \frac{1-e^{-t}}{t}$ in the formula (5) and using the limit

$$\lim_{t\to 0} u = \lim_{t\to 0} h(t) = \lim_{t\to 0} \frac{1-e^{-t}}{t} = 1$$

yield

$$\frac{\mathrm{d}^k}{\mathrm{d}\,t^k}\left[\left(\frac{t}{1-e^{-t}}\right)^{x+1}\right]=\sum_{\ell=0}^{k}\frac{\langle-(x+1)\rangle_\ell}{u^{(x+1)+\ell}}\mathrm{B}_{k,\ell}\big(h'(t),h''(t),\dots,h^{(k-\ell+1)}(t)\big)$$

$$\to\sum_{\ell=0}^{k}\langle-(x+1)\rangle_\ell\mathrm{B}_{k,\ell}\big(h'(0),h''(0),\dots,h^{(k-\ell+1)}(0)\big)$$

as $t\to0$, where

$$\langle x\rangle_n=\prod_{\ell=0}^{n-1}(x-\ell)=\begin{cases}x(x-1)\cdots(x-n+1), & n\ge1\\ 1, & n=0\end{cases}$$

is the falling factorial of $x\in\mathbb{R}$ for $n\in\{0\}\cup\mathbb{N}$. It is not difficult to see that

$$\langle-(x+1)\rangle_\ell=\prod_{m=0}^{\ell-1}[-(x+1)-m]=(-1)^\ell\prod_{m=0}^{\ell-1}(x+m+1)$$

$$=(-1)^\ell\prod_{m=1}^{\ell}(x+m)=\frac{(-1)^\ell}{x}\prod_{m=0}^{\ell}(x+m)=\frac{(-1)^\ell}{x}(x)_{\ell+1},$$

where $(x)_n$ is defined by (1). Hence, it follows that

$$\langle-(x+1)\rangle_\ell=\frac{(-1)^\ell}{x}\sum_{m=0}^{\ell+1}s(\ell+1,m)x^m,\quad \ell\ge0.$$

Since

$$h^{(\ell)}(t)=\int_{1/e}^{1}s^{t-1}(\ln s)^\ell\,\mathrm{d}\,s\to\frac{(-1)^\ell}{1+\ell},\quad t\to0,\quad \ell\ge0,$$

by virtue of (3) and (4), we have

$$\mathrm{B}_{k,\ell}\big(h'(0),h''(0),\dots,h^{(k-\ell+1)}(0)\big)=\mathrm{B}_{k,\ell}\left(-\frac{1}{2},\frac{1}{3},\dots,\frac{(-1)^{k-\ell+1}}{k-\ell+2}\right)$$

$$=(-1)^k\mathrm{B}_{k,\ell}\left(\frac{1}{2},\frac{1}{3},\dots,\frac{1}{k-\ell+2}\right)=\frac{(-1)^{k+\ell}k!}{(k+\ell)!}\sum_{i=0}^{\ell}(-1)^i\binom{k+\ell}{\ell-i}S(k+i,i).$$

Accordingly, we obtain

$$\begin{aligned}\lim_{t\to0}\frac{\mathrm{d}^k}{\mathrm{d}\,t^k}\left[\left(\frac{t}{1-e^{-t}}\right)^{x+1}\right]\\ &=\sum_{\ell=0}^{k}\left[\frac{(-1)^\ell}{x}\sum_{m=0}^{\ell+1}s(\ell+1,m)x^m\right]\left[\frac{(-1)^{k+\ell}k!}{(k+\ell)!}\sum_{i=0}^{\ell}(-1)^i\binom{k+\ell}{\ell-i}S(k+i,i)\right]\\ &=\frac{(-1)^k k!}{x}\sum_{\ell=0}^{k}\frac{1}{(k+\ell)!}\left[\sum_{i=0}^{\ell}(-1)^i\binom{k+\ell}{\ell-i}S(k+i,i)\right]\left[\sum_{m=0}^{\ell+1}s(\ell+1,m)x^m\right]\\ &=\frac{(-1)^k k!}{x}\sum_{m=0}^{k+1}\left[\sum_{\ell=m-1}^{k}\frac{s(\ell+1,m)}{(k+\ell)!}\sum_{i=0}^{\ell}(-1)^i\binom{k+\ell}{\ell-i}S(k+i,i)\right]x^m.\end{aligned}$$

The explicit formula (2) is thus proved. The proof of Theorem 1 is complete.

Remark 1. This paper is a slightly revised version of the preprint [13].

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